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On the Origin of Turbulence

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We are all familiar with the fact that a linear flow of water in a tube can be obtained only for velocities below a certain critical limit and that, when the velocity exceeds this limit, laminar flow ceases and a complex, irregular, and fluctuating motion sets in. More generally than in this context of flow through a tube, it is known that motions governed by the equations of Stokes and Navier change into turbulent motion when a certain nondimensional constant called the Reynolds number exceeds a certain value of the order of 1000 (P. Bradshaw, 1978). This Reynolds number depends upon the linear dimension, L, of the system, the coefficient of viscosity μ, the density ρ, and the velocity v in the following manner $R = \frac{\rho v L}{\mu}$. Following (S. Chandrasekhar (1949), ApJ **110**, 329) we can make us the question: What is the reason that a phenomenon like turbulence can occur at all?. We describe the turbulence in fluids as a consequence of the inherent discontinuity of matter. We start with the description of matter density as a discontinuous Dirichlet integral function, and through the Euler equation for matter conservation, we obtain a differential equation which implies a transference of velocity (and then energy) from one eddys to others, i.e. from one scale to another, which is one of the main observational features of turbulence.

KEY WORDS: turbulence.
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1. INTRODUCTION

The answer to the Chandrasekhar's question could be that an ideal fluid is a mechanical system with a very large number of degrees of freedom and that, in consequence, it is theoretical capable of a very large number of different types of motions. Laminar motion is only one of the many possible motions that the system is capable of. It is far more likely that the possible motions will be simultaneously present. The fundamental problem of turbulence would therefore appear to be a statistical one of specifying the probability with which the various types of motion may occur and are present. It is clear that the problem of turbulence has an analogy with the problem of analyzing a continuous spectrum of radiation. In the latter case, the greatest interest is generally attached to the distribution of intensity in the spectrum and only secondary to the phase relationships.

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Considering the state of motion at a given instant, we may analyze the fluctuating velocity field as a result of superposition of periodic variations with all possible wave lengths. We may picture the component with a wave length λ as corresponding to an eddy of size λ, and, since many wave lengths are needed to represent a general velocity field, we may speak of a hierarchy of eddies. This hierarchy of eddies will be limited on the side of long wave lengths by the fact that no eddy of size larger than the dimension of the medium in which we analyze the turbulence can occur. Instead of the wave length λ, it is often more convenient to speak of a wave number $k = \frac{2\pi}{\lambda}$.

We know that under conditions of equilibrium the distribution of energy in the continuous spectrum will be that given by Planck's law. We may ask whether a similar equilibrium spectrum exists for turbulence. In answering this question, we must keep in mind one important distinction between the optical analogue and turbulence. In the optical case the equilibrium Planck spectrum will be reached, no matter what the initial distribution is. In contrast, turbulence can be maintained only by an external agency, like continuous stirring, the energy available from thermal instability, or rotation in a differentially rotating atmosphere. In other words, energy is required for maintenance of turbulence; in the absence of such an agency, turbulence will decay. We may suppose that the energy supplied by the external agency is communicated principally to the largest eddies, then the energy is being dissipated by viscosity into thermal energy, effected principally by the smallest eddies, in which the motions may be expected to be laminar. So, energy must flow through the entire hierarchy of eddies.

Theories of turbulence fall into three categories: the heuristic theories which attempt to describe turbulence in terms of certain "a priori" concepts (such as mean free path and eddy viscosity) derived from the kinetic theory of gases but which are not deducible from the equations of motion (Heisenberg, 1948a,b); the phenomenological theories which derive certain relations which must obtain in virtue of the equations of motion and continuity on certain well-defined hypotheses (such as homogeneity and isotropy) (Taylor, 1935, 1938); and deductive physical theories (Chandrasekhar, 1955; Saveliev and Gorokhovski, 2005) which attempt to predict the time evolution of correlations in the velocity components. All of them are statistical theories. In the present paper we obtain a deterministic differential equation which implies a transference of energy between the eddies, which is one of the main observational features of turbulence.

2. THE MODEL

If we get a new step in the approaching the real nature of matter we must be able of describe the observation that the matter is discontinuous in its essence. For example we know that more than the 99% of an atom is without material particles, i.e. without protons, neutrons or electrons. Here we take, as a mathematic model to describe the matter, a functional form for density of matter like the well known Dirichlet integral, modified to describe the density as a function of space and time:

$$\rho(x,t) = \frac{1}{\pi}\Sigma_k \int \frac{\sin(x_k t)}{t} e^{it(x+k)}\, dt \tag{1}$$

then we have $\rho(x,t) = 1$ if $-x_k \leq x + k \leq x_k$ and $\rho(x,t) = 0$ in any other case, where k describe the central position of real matter structures (e.g. atomic nucleus), and x_k means the size of these structures. These structures will depend on each problem.

Now, although one know that the velocity of matter must be obtained from the forces equillibrium differential equation, and due to here we are dealing only with the intrinsic behavior of matter, we consider the continuity equation:

$$\frac{\partial \rho}{\partial t} = -\overrightarrow{\nabla}(\rho \vec{v}) \tag{2}$$

and introducing Eq. (1) into Eq. (2), considering one dimension and assuming a velocity which is $\frac{\partial v_x}{\partial x} \simeq 0$, we have

$$v_x = -\Sigma_k \frac{\sin(x_k t)}{it^2} e^{it(x+k)} + \text{const.} \tag{3}$$

if $-x_k \leq x + k \leq x_k$.

Plotting v_x againsnt time (see Fig. 1) one can see the characteristic observational behavior of turbulence (there is no any oscillation equal to another one) decaying because there is no any energy supply.

On the other hand, if one compare Eq. (3) with the classic decomposition of velocities in turbulence at different scales $v(x,t) = \Sigma_k v_k(t) e^{ikx}$, and taking $t = x/c$, with c being the light speed, one has

$$v_k = \frac{\sin(x_k x/c)}{i\frac{x^2}{c^2}} e^{\frac{ix^2}{c}} \tag{4}$$

and so

$$\frac{v_k}{v_{k+n}} = \frac{\sin(x_k t)}{\sin(x_{k+n} t)} \tag{5}$$

which is a relation (to be contrasted with measurements, i.e. it is a prediction) between each eddy's velocity with all of the other eddy's velocities. Moreover, taking partial derivatives of Eq. (3) with respect to t and k and taking only the real parts, one obtain the relation

$$\frac{x+k}{t}\frac{\partial v_k}{\partial k} = \frac{\partial v_x}{\partial t} \tag{6}$$

and due to $-x_k \leq x + k \leq x_k$ there will be negative values for $x + k$, then implying a transference of v_x between different k's as the time run, like some kind of

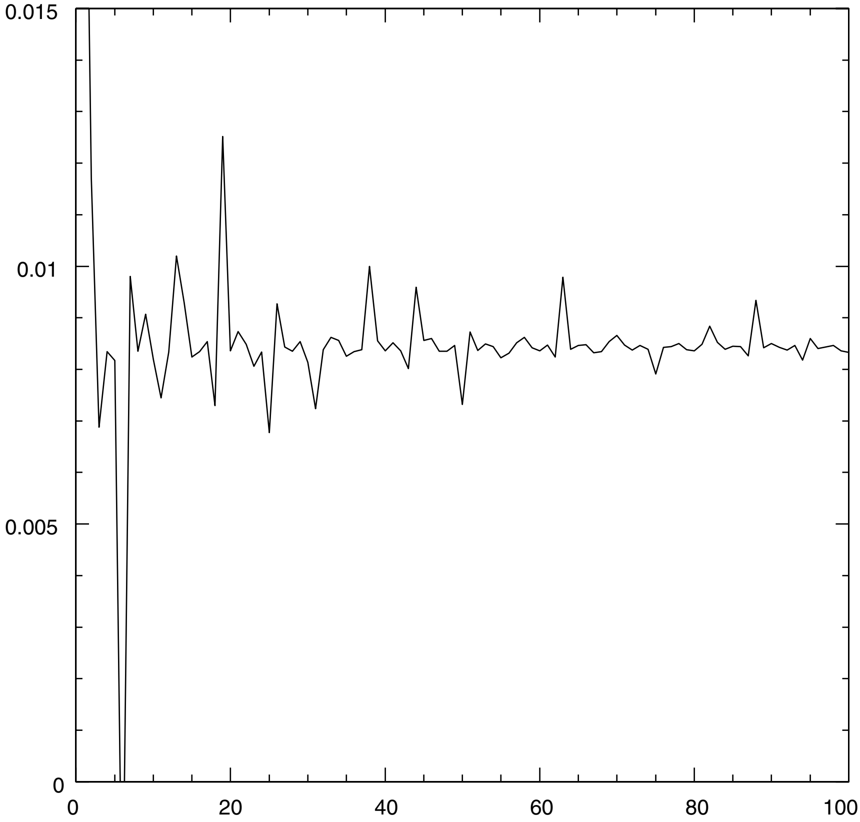

Fig. 1. Turbulent velocity against time as is obtained from equation (3). One can see the characteristic observed behavior of not any oscillation equal to another one, so as the expected decaying because there is no any energy supply.

continuity equation similar to Eq. (2) for density of matter. So we can explain the main observational feature of turbulence, i.e. the transference of energy ($\sim v_k^2$) from one eddy at scale k to another at different scale $k + n$ for all n.

Moreover, if one apply some external force to the system, the result is as expected by real observations. So, if one take a sinusoidal expression for the force added and assume coupling of frequencies (internal and external) after some characteristic relaxation time, we have an expression for the final turbulent velocity like:

$$v_x = -\Sigma_k \frac{\sin(x_k t)}{it^2} e^{it(x+k)} + h_1 \ \sin(t(x+k)) + \text{const.} \tag{7}$$

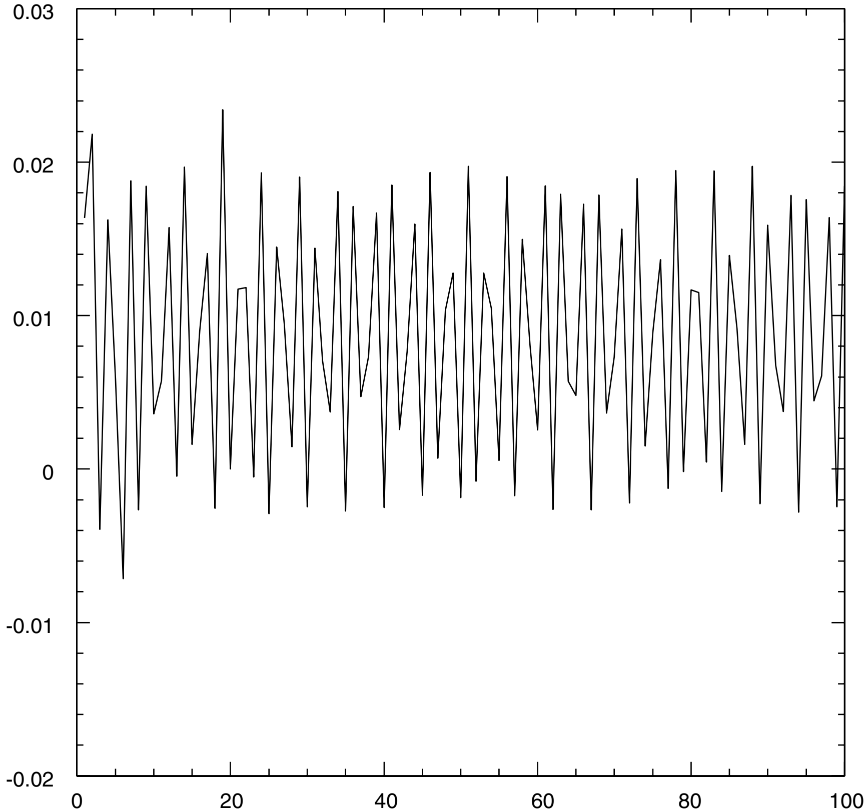

Fig. 2. Turbulent velocity against time as is obtained from Eq. (7). One can see the characteristic observed behavior of not any oscillation equal to another one, so as the observational maintained oscilations due to the supply of some external force.

if $-x_k \leq x + k \leq x_k$, with h_1 being some normalization constant. In Fig. 2 we have plotted velocity v. time, and we can see the non-regular oscillations characteristic of turbulent behaviour.

3. CONCLUSIONS

We have described the turbulence in fluids as a consequence of the inherent discontinuity of matter. We started with the description of matter density as a discontinuous Dirichlet integral function, and through the Euler equation for matter conservation, we obtained a differential equation which implies a transference of velocity (and then energy) from one eddies to others, i.e. from one scale to another, which is one of the main observational features of turbulence.

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